Nonminimal Derivative Coupling and the Recovering of Cosmological Constant

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Abstract

We show that the existence of the cosmological constant can be connected to a nonminimal derivative coupling, in the action of gravity, between the geometry and the kinetic part of a given scalar field without introducing any effective potential of scalar fields. Exact solutions are given.

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1 Introduction

Including nonlinear terms of the various curvature tensors (Riemann, Ricci, Weyl) and nonminimally coupled terms in the effective action of gravity has become, recently, a very common trend from quantum field theory side and cosmology [1]. The basic motivation for studying such theories comes from the fact that they provide a possible approach to quantum gravity from a perturbative point of view. Furthermore they occur as low–energy limit of several unification scheme as, *e.g.* superstring theory [2].

Modern cosmology, starting from the pioneering works by Starobinsky [3], has found into them a fruitful arena for trying to solve the several shortcomings of standard cosmological model as initial singularity, flatness, horizon problems and so on, in the framework of the inflationary paradigm.

In fact, nonminimal coupling between scalar field(s) and geometry and higher order terms in the curvature invariants naturally give rise to inflationary solutions which, in various senses, improve the early inflationary models (see for example [4],[5]).

Another important question connected to such theories is a dynamical determination of cosmological constant which could furnish the gravity vacuum state [6] and could contribute to solve the dark matter problem: in fact, the presence of a cosmological constant gives rise to viable models for large scale structure [7] as recent observations are confirming [8].

Besides, the exact determination of cosmological constant could account for the fate of the whole Universe considering the so called no–hair conjecture [9]. In any case, we need a time variation of cosmological constant to satisfy issues as successful inflationary models, the agreement with large–scale structure observations, and to obtain a de Sitter stage in the future, if a *remnant* of cosmological constant is present into the overall dynamics.

In other words, the cosmological constant should have acquired high values at early times (de Sitter stage), should have undergone a phase transition with a graceful exit (to recover the observed dust dominated Friedman stage) and should result in a remnant in the future.

Considering the wide variety of extended gravity theories which can give rise to de Sitter stages (*i.e.* where cosmological constant leads dynamics) a main question is to recover classes of gravitational theories which "naturally" give rise to cosmological constant without putting it "by hand" or considering special initial data [10]. Furthermore, is cosmological constant related only with the presence of an effective potential of a scalar field? Can it be recovered also by introducing nonminimal couplings between geometry and scalar fields without introducing any sort of potential?

We have to stress the fact that also considering pure higher order theories as $f(R) = R + \alpha R^2 + \ldots$, by a conformal transformation higher order terms give rive effective potential. In general, these theories have de Sitter stages [11],[12].

In this paper we want to show that it is possible to recover the de Sitter behaviour, and then the cosmological constant by introducing nonminimal derivative couplings between

the geometry and the scalar field.

These kind of couplings naturally arise in Kaluza–Klein and superstring theories [13] while they must be included in the matter Lagrangian of quantum field theories in curved spacetimes involving scalar fields and a multiloop expansion [14].

From a general point of view the effective Lagrangian of quantum gravity, considering the expansion of curvature invariants and the matter fields, can be written as [15]

$$\mathcal{L}_{effective} = \mathcal{L}_g + \mathcal{L}_m \,, \tag{1}$$

where

$$\mathcal{L}_g = \sqrt{-g} \left\{ \Lambda - \frac{R}{2} + a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + \mathcal{O}(R^3) \right\} \,, \tag{2}$$

and

$$\mathcal{L}_m = \sqrt{-g} \left\{ \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + \right. \tag{3}$$

$$\left. + d_1 R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + R(d_2 \partial_\mu \phi \partial^\mu \phi + d_3 m^2 \phi^2) + \ldots \right\} \,.$$

We are assuming $8\pi G = 1$; a_i and d_i are coupling constants which scale as powers of the mass. Here the gravitational Lagrangian has been ordered in a derivative expansion of the metric with Λ being of order ∂^0, R of the order ∂^2, R^2 and $R_{\mu\nu} R^{\mu\nu}$ of order ∂^4, and so on.

In four dimensions, we do not need to include terms as $R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$ in the action since, by the Gauss–Bonnet theorem, it is possible to express these terms as R^2 and $R_{\mu\nu} R^{\mu\nu}$. Several papers have been devoted to the R^2 and scalar field cosmologies but not so much have pointed out the relevance of derivative coupling in order to recover the cosmological constant. In [16], for example, a systematic study of phase space and inflationary attractors is done for this kind of cosmologies. However, in our knowledge, it has never been stressed how it is possible to recover "exactly" de Sitter behaviours and cosmological constant starting from them.

This issue takes relevance in the debate of how to recover the vacuum state in general relativity. If the de Sitter stage is obtained without considering effective scalar fields potential as $V(\phi) \simeq (\lambda/n)\phi^n$ or $V(\phi) \simeq m^2 \phi^2$ (or the conformal transformed field potential starting from theories as R^2), it means, in our opinion, that these terms are not so essential for recovering the cosmological constant (in this case from the matter field side).

In Sec.2, we consider an action where the nonminimal derivative coupling is introduced in a simple way. We show that de Sitter solutions exists. Sec.3 is devoted to the study of an action in which nonminimal coupling is introduced for the field ϕ and for its covariant derivative $\phi_\mu \equiv \nabla_\mu \phi$. A general discussion on how it is possible to construct an effective cosmological constant and how to recover asymptotically a de Sitter behaviour is done in Sec.4. We follow the method outlined in [10],[18]. Conclusions are drawn in Sec.5.

2 Minimal Coupling with Nonminimal Derivative Coupling

Let us start our considerations from the action

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[-\frac{R}{2} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \zeta R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi R g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] . \tag{4}$$

We have not introduced any effective scalar field potential. In a Friedman–Robertson–Walker (FRW) metric, the action (4) reduces to the form

$$\mathcal{A} = 2\pi^2 \int dt a^3 \left\{ \left[\left(\frac{\ddot{a}}{a} \right) + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] (3 - 6\xi\dot{\phi}^2) + \frac{\dot{\phi}^2}{2} - 3\zeta \left(\frac{\ddot{a}}{a} \right) \dot{\phi}^2 \right\} . \tag{5}$$

Integrating by parts and eliminating the boundary terms, we get the pointlike Lagrangian

$$\mathcal{L} = 3a\dot{a}^2(1 + \eta\psi^2) + 6\chi a^2 \dot{a}\psi\dot{\psi} - \frac{1}{2}a^3\psi^2 , \tag{6}$$

where

$$\eta = -2(\xi + \zeta), \qquad \chi = -(2\xi + \zeta) , \tag{7}$$

and we are considering, for simplicity, the spatially flat case ($k = 0$). To reduce the degree of the derivative term of scalar field, we define the auxiliary field

$$\psi = \dot{\phi} , \tag{8}$$

so that the Lagrangian (6) assumes the canonical form $\mathcal{L} = \mathcal{L}(a, \dot{a}, \psi, \dot{\psi})$. The equations of motion are

$$(2\dot{H} + 3H^2)(1 + \eta\psi^2) + 4\eta H\psi\dot{\psi} + 2\chi\dot{\psi}^2 + 2\chi\psi\ddot{\psi} + \frac{\psi^2}{2} = 0 , \tag{9}$$

$$6\chi(\dot{H} + 3H^2) = 6\eta H^2 - 1 , \tag{10}$$

$$3H^2(1 + \eta\psi^2) + 6\chi H\psi\dot{\psi} + \frac{\psi^2}{2} = 0 , \tag{11}$$

where $H = \dot{a}/a$ is the Hubble parameter. Immediately we see that the particular solution

$$\dot{\psi} = 0 \rightarrow \psi = \psi_0, \quad H^2 = \frac{\Lambda}{3} , \tag{12}$$

which is de Sitter, exists and

$$\Lambda = \frac{1}{2(4\xi + \zeta)}, \quad \psi_0 = \frac{1}{\sqrt{\zeta - 2\xi}} . \tag{13}$$

The cosmological behaviour is then given by

$$a(t) = a_0 \exp \sqrt{\frac{\Lambda}{3}} t , \tag{14}$$

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$$\phi(t) = \psi_0 t \, . \tag{15}$$

This result tells that a cosmological constant can be constructed by the parameters of nonminimal derivative coupling. The general solution of system (9)–(11) is obtained taking into account (10) which can be recast as

$$\dot{H} = AH^2 + B \tag{16}$$

where

$$A = \frac{\eta - 3\chi}{\chi}, \qquad B = -\frac{1}{6\chi} \, . \tag{17}$$

Two interesting sub–cases, due to the sign of B/A, can be discussed:
i) $B/A > 0$ implies the solution

$$H(t) = \sqrt{\frac{B}{A}} \tan \sqrt{AB} \, (t - t_0) \, . \tag{18}$$

ii) If $B/A < 0$, it follows that $H^2 > |B/A|$ and then

$$H = \sqrt{\frac{|B|}{|A|}} \left[\frac{1 + \exp 2\sqrt{|A||B|} \, (t - t_0)}{1 - \exp 2\sqrt{|A||B|} \, (t - t_0)} \right] , \tag{19}$$

or $H^2 < |B/A|$, so that

$$H = \sqrt{\frac{|B|}{|A|}} \left[\frac{\exp 2\sqrt{|A||B|} \, (t - t_0) - 1}{\exp 2\sqrt{|A||B|} \, (t - t_0) + 1} \right] . \tag{20}$$

In both cases we recover asymptotically the solution (12) and (13).

The time evolution of ψ (and then of ϕ) is obtained introducing these first integrals into Eq.(9) or Eq. (11). However Eq.(11) is the energy condition $E_{\mathcal{L}} = 0$ (*i.e.* the $(0,0)$ Einstein equation) which gives the constraints on the initial conditions. Integrating Eq.(19) we get

$$a(t) = a_0 \left\{ \frac{\exp 2\sqrt{|A||B|} \, t}{\left[\exp 2\sqrt{|A||B|} \, t - 1 \right]^2} \right\}^{1/2|A|} . \tag{21}$$

From Eq.(20)

$$a(t) = a_0 \left\{ \frac{\left[\exp 2\sqrt{|A||B|} \, t + 1 \right]^2}{\exp 2\sqrt{|A||B|} t} \right\}^{1/2|A|} . \tag{22}$$

Asymptotically we can get increasing and decreasing exponential behaviour. However, the first case is of physical interest. Eq. (18) gives an oscillatory behaviour for the scale factor $a(t)$. All these solutions are parameterized by the derivative coupling ζ and ξ.

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3   Nonminimal Couplings for Scalar Field and its Derivative

The generalization of above considerations can be obtained by introducing a nonminimal coupling also in the standard part of gravitational Lagrangian. The effective action becomes

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \zeta R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi R g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right] . \tag{23}$$

The most general action should involve terms as

$$h(\phi)R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi , \quad g(\phi)R g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi , \tag{24}$$

but, for the sake of simplicity, we restrict to the parameters ζ and ξ. However, the standard Newtonian coupling is recovered for $F(\phi) = -1/2$. Using the above procedure, the pointlike FRW Lagrangian is

$$\mathcal{L} = 6Fa\dot{a}^2 + 6F'a^2\dot{a}\dot{\phi} - 6Fka + \frac{1}{2}a^3\dot{\phi}^2 + \tag{25}$$

$$+ 6\zeta(a\dot{a}^2\dot{\phi}^2 + a^2\dot{a}\dot{\phi}\ddot{\phi}) + 6\xi(a\dot{a}^2\dot{\phi}^2 + 2a^2\dot{a}\dot{\phi}\ddot{\phi} - ka\dot{\phi}^2) .$$

where the prime represents the derivative with respect the scalar filed ϕ. In order to make it canonical, we have to impose

$$\zeta = -2\xi . \tag{26}$$

Then, Eq.(25) reads

$$\mathcal{L} = 6Fa\dot{a}^2 + 6F'a^2\dot{a}\dot{\phi} + \frac{1}{2}a^3\dot{\phi}^2 - 6\xi a\dot{a}^2\dot{\phi}^2 , \tag{27}$$

choosing the spatially flat model $k = 0$. The corresponding equations of motion are

$$\ddot{\phi} + 3H\dot{\phi} + 6(\dot{H} + 2H)F' - 12\xi(2H\dot{H}\dot{\phi} + 3H^3\dot{\phi} + H^2\ddot{\phi}) = 0 , \tag{28}$$

$$(2\dot{H} + 3H^2)F + 2\dot{F}H + \ddot{F} - \xi(2\dot{H} + 3H^2)\dot{\phi}^2 - 4\xi H\dot{\phi}\ddot{\phi} - \frac{\dot{\phi}^2}{4} = 0 , \tag{29}$$

$$H^2F + H\dot{F} + \frac{1}{12}\dot{\phi}^2 - 3\xi H^2\dot{\phi}^2 = 0 . \tag{30}$$

The form of the solutions strictly depends on the form of the coupling $F(\phi)$. The existence of the de Sitter solution can be imposed so that the form of the coupling is determined as a function $F(\phi(t))$ (see also [17]). Assuming, in general,

$$a(t) = a_0 e^{\lambda t} , \tag{31}$$

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where λ is a constant (not necessarily the "right" cosmological constant) we can derive an equation for $F(\phi(t))$ from (28)–(30). With a little algebra, we get

$$\ddot{F} + c_1 \dot{F} + c_2 F = 0 \,, \tag{32}$$

where the constants $c_{1,2}$ are defined as

$$c_1 = \frac{(12\xi\lambda^2 - 5)\lambda}{(12\xi\lambda^2 - 1)}, \quad c_2 = 6\lambda^2 \,. \tag{33}$$

It follows that the coupling $F(t)$ and the scalar field $\phi(t)$ are

$$F(t) = F_1 e^{\alpha_1 t} + F_2 e^{\alpha_2 t} \,, \tag{34}$$

$$\phi(t) = \phi_0 \pm \sigma_1 \int \sqrt{b_1 e^{\alpha_1 t} + b_2 e^{\alpha_2 t}} \, dt \tag{35}$$

where

$$\alpha_{1,2} = -\frac{c_1}{2} \pm \sqrt{\left(\frac{c_1}{2}\right)^2 - c_2} \,, \tag{36}$$

and

$$\sigma_1 = \frac{1}{\sqrt{3\xi\lambda^2 - 1/12}} \,. \tag{37}$$

$F_{1,2}$ and $b_{1,2}$ are integration constants. Asymptotically, we have $F(t) \to F_2 e^{\alpha_2 t}$ and then

$$F(\phi) \sim F_0 (\phi - \phi_0)^2 \,, \tag{38}$$

where F_0 depends on λ and ξ. Also in this case the existence of cosmological constant strictly depends on the derivative coupling while it can be shown that it does not exist if only nonderivative nonminimal coupling is present [10]. In other words, for a pure Brans–Dicke theory it is not possible to recover a cosmological constant unless a scalar field potential is introduced by hands.

Finally, it is last term in the Lagrangian (27) which plays the fundamental role to obtain the de Sitter behaviour.

4 General Considerations and the Effective Cosmological Constant

Following [10],[12],[18], [19], it is possible to show that for several extended gravity models, a de Sitter behaviour can be recovered. This means that Wald's proof of no–hair theorem [20] can be extended without putting "by hands" any cosmological constant but recovering it by general considerations and defining an "effective" cosmological constant.

The proof of this statement can be easily sketched following the arguments in [10] and [18]. An asymptotic cosmological constant is recovered any time that the asymptotic conditions

$$(H - \Lambda_1)(H - \Lambda_2) \geq 0 \,, \tag{39}$$

$$\dot{H} \leq 0 \,, \tag{40}$$

hold. However $\Lambda_{1,2}$ are constants. In our case, considering Eq.(30), we can define an effective cosmological constant as

$$\Lambda_{eff\,1,2} = \frac{1}{2(F - 3\xi\dot{\phi}^2)} \left[-\dot{F} \pm \sqrt{\dot{F}^2 - \frac{\dot{\phi}^2}{3}(F - 3\xi\dot{\phi}^2)} \right] \,, \tag{41}$$

so that, Eq.(30) can be recast in the form

$$(H - \Lambda_{eff,1})(H - \Lambda_{eff,2}) = 0 \,. \tag{42}$$

For the physical consistency of the problem, it has to be

$$\dot{F}^2 - \frac{\dot{\phi}^2}{3}(F - 3\xi\dot{\phi}^2) \geq 0 \,, \tag{43}$$

or simply

$$F - 3\xi\dot{\phi}^2 < 0 \,. \tag{44}$$

The asymptotic cosmological constant is recovered if the conditions

$$\frac{\dot{F}}{F - 3\xi\dot{\phi}^2} \longrightarrow \Sigma_0 \,, \tag{45}$$

$$\frac{\dot{\phi}^2}{F - 3\xi\dot{\phi}^2} \longrightarrow \Sigma_1 \,, \tag{46}$$

hold. $\Sigma_{0,1}$ are constants which determine the asymptotic behaviour of F and ϕ (see also [18]).

Considering Eq.(29), we have

$$\dot{H} = -\frac{3}{2}H^2 - \frac{H}{F - \xi\dot{\phi}^2}\frac{d}{dt}(F - \xi\dot{\phi}^2) - \frac{\ddot{F}}{2(F - \xi\dot{\phi}^2)} + \frac{\dot{\phi}^2}{8(F - \xi\dot{\phi}^2)} \,. \tag{47}$$

The theorem is easily shown if

$$\frac{d}{dt}(F - \xi\dot{\phi}^2) < 0 \tag{48}$$

and

$$\ddot{F} < 0 \,, \tag{49}$$

which are very natural conditions which restrict the possible form of F. Furthermore, it must be $F < 0$ in order to recover attractive gravity (the standard coupling is for $F \rightarrow -1/2$). We stress again the relevant role played by the derivative coupling in the construction of cosmological constant.

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5 Conclusion

In this paper, we have outlines the role of nonminimal derivative coupling in recovering the de Sitter behaviour and then the cosmological constant. The main point is the fact that we have not used any effective potential but the cosmological constant is strictly related to the derivative coupling and, as it is shown in [10], it cannot be recovered if the coupling is not derivative. Also by conformal transformation [16], it is possible to show that the right–hand side of Einstein equations cannot be written as a scalar field energy–momentum tensor in its proper meaning.

In conclusion, the cosmological constant can be reconstructed, at least at a classical level, by the kinetic part of an intervening scalar field "without" considering scalar field potential. In this sense, we can deal with a "dynamical cosmological constant" which could come out from the interactions between scalar field matter and geometry. A further step in this analysis is to consider more generic derivative couplings and effective potentials in (23) in order to see what is the specific role of all ingredients in the "construction" of the effective cosmological constant.

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